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Convergence of Padé Approximants for a Certain Class of Meromorphic Functions

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In [1] we considered the convergence of diagonal sequences of Padé approximants of meromorphic functions of the special type

$$f(x) = \sum_{i=1}^{\infty} \frac{A_i}{a_i - x}, \tag{1}$$

where the A_i and a_i are complex numbers satisfying

$$\sum_{i=1}^{\infty} \left| \frac{A_i}{a_i} \right| < \infty.$$

The method utilized certain series expansions for the persymmetric determinants $\Delta_{m-1,n} = | c_{n+i-j} |_{i,j=0}^{m-1}$, namely,

$$\Delta_{m-1,n} = (-1)^{m(m-1)/2} \sum_{k_1 < k_2 < \cdots < k_m} A_{k_1} A_{k_2} \cdots A_{k_m} (a_{k_1} a_{k_2} \cdots a_{k_m})^{-(n+m)} \times \prod_{1 \leqslant i < j \leqslant m} (a_{k_j} - a_{k_i})^2,$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

and so

$$c_n = \sum_{i=0}^{\infty} \frac{A_i}{a_i^{n+1}}.$$

We use this expansion below to give an elementary treatment of the convergence of diagonal sequences of Padé approximants of functions of the form

$$f(x) = c_0 + c_1 + \cdots + c_{r-1}x^{r-1} + \sum_{i=1}^{\infty} A_i \left[\frac{1}{a_i - x} - \frac{1}{a_i} - \frac{x}{a_i^2} - \cdots - \frac{x^{r-1}}{a_i^r}\right] \tag{2}$$

satisfying the conditions:

(i) the c_i ; $i = 0, 1,..., r - 1$ are arbitrary complex numbers,

(ii) the A_i are real,

(iii) the a_i lie on a line L through the origin, i.e., for some θ, $a_i = d_i e^{i\theta}$ for all i with the d_i real, and

(iv) $0 < | d_1 | < d_2 | < d_3 | < \cdots$ with $\sum_{i=1}^{\infty} | A_i/d_i^{r+1} | < \infty$.

When $r = 0$, we take $f(x)$ to be of the form (1). Let S_θ denote the part of the line L given by

$$S_\theta = \begin{cases} \{x : \arg x = \theta, | x | \geqslant d_1\} & \text{if } d_i > 0 \quad \text{for all } i, \\ \{x : x \in L, | x | \geqslant | d_1 |\} & \text{if } d_i < 0 \quad \text{for some } i. \end{cases}$$

We let $f_{m,n}(x) = P_{m,n}(x)/Q_{m,n}(x)$ denote the reduced (m, n)-Padé approximant of $f(x)$. We have the following theorem.

THEOREM. *Let $f(x)$ be as described in (2). If p is an integer such that $p \geqslant r - 1$ and $A_i/d_i^p > 0$ for all i, then the sequence $\{f_{m,m+p}(x)\}$ converges to $f(x)$ uniformly on any compact set bounded away from S_θ.*

Remark. When $r = 0$ and $p = -1$ we interpret the sequence of approximants as the diagonal file lying immediately below the principal diagonal file of the Padé table.

The theorem covers three cases.

Case A. If $A_i > 0$ and $d_i > 0$ for all i, we may take $p \geqslant r - 1$ to be *arbitrary.*

Case B. If $A_i > 0$ for all i, but $d_i < 0$ for some i, we take $p \geqslant r - 1$ to be *even.*

Case C. If $\operatorname{sign} A_i = \operatorname{sign} d_i$ for all i, we take $p \geqslant r - 1$ to be *odd.*

To facilitate the exposition of the proof of the theorem we list some notation and formulas which were given in [1] in slightly modified form:

(3) $\Delta_{m,n}^{(i)}$ and $\Delta_{m,n}(x)$ are the determinants obtained from $\Delta_{m,n}$ by replacing the first row by $c_{n+m+i}, c_{n+m+i-1},..., c_{n+i}$ and by $1, x, x^2,..., x^m$, respectively.

(4) Let $S(m)$ denote the set of all m-tuples $(k_1, k_2, ..., k_m)$ of positive integers such that $k_1 < k_2 < \cdots < k_m$. For every $k \in S(m)$, define

$$T^k_{m,m+p} = A_{k_1} A_{k_2} \cdots A_{k_m} (d_{k_1} d_{k_2} \cdots d_{k_m})^{-(2m+p)} \prod_{1 \leqslant i < j \leqslant m} (d_{k_j} - d_{k_i})^2.$$

(5) For $p \geqslant r - 1$,

$$\Delta_{m-1,m+p} = (-1)^{m(m-1)/2} e^{-m(m+p+1)i\theta} \sum_{k \in S(m)} T^k_{m,m+p}$$

(6) If $p \geqslant r - 1$,

$$\Delta_{m,m+p}(x) = (-1)^{m(m-1)/2} e^{-m(m+p+1)i\theta} \sum_{k \in S(m)} T^k_{m,m+p} \prod_{\nu=1}^{m} \left(1 - \frac{x}{a_{k_\nu}}\right).$$

(7) For $\Delta_{m-1,m+p} \neq 0$,

$$Q_{m,m+p}(x) = \Delta_{m,m+p}(x)/\Delta_{m-1,m+p}.$$

(8) If $\Delta_{m-1,m+p} \neq 0$,

$$f(x)\, Q_{m,m+p}(x) - P_{m,m+p}(x) = \Delta^{-1}_{m-1,m+p} \sum_{i=1}^{\infty} \Delta^{(i)}_{m,m+p} x^{2m+p+i}.$$

(9) If $\Delta_{m-1,n} \neq 0$ and $n \geqslant m - 1$,

$$P_{m+1,n+1} Q_{m,n} - P_{m,n} Q_{m+1,n+1} = (-1)^m (\Delta_{m,n+1}/\Delta_{m-1,n})\, x^{m+n+1}.$$

(10)
$$\frac{P_{m,m+p}}{Q_{m,m+p}} = \frac{P_{0,p}}{Q_{0,p}} + \sum_{i=0}^{m-1} (-1)^i \frac{\Delta_{i,i+p+1}}{\Delta_{i-1,i+p}} \frac{x^{2i+p+1}}{Q_{i,i+p} Q_{i+1,i+p+1}}$$

if $\Delta_{i-1,i+p} \neq 0$ for $i = 0, 1, ..., m - 1$.

The expansions (5) and (6) were originally given only in the case where $r = 0$ and $\theta = 0$. That they remain valid for functions of the form (2), when $p \geqslant r - 1$, is easily seen from the fact that the determinants $\Delta_{m-1,m+p}$ and $\Delta_{m,m+p}(x)$ involve only the c_n for $n \geqslant r$ and we have for $n \geqslant r$ that

$$c_n = \sum_{i=1}^{\infty} \frac{A_i}{a_i^{n+1}}.$$

The formulas then follow as in [1].

In order to discuss the convergence of the sum (10) we will need appropriate estimates for the quantities $| \Delta_{i,i+p+1}/\Delta_{i-1,i+p} |$ and $| Q_{i,i+p}(x)|$. These are obtained in the following lemmas.

LEMMA 1. *Under the conditions of the theorem,* $\Delta_{m-1,m+p} \neq 0$ *for all* m *and*

$$| \Delta_{m,m+p+1}/\Delta_{m-1,m+p} | \leqslant | c_{p+1} | \, | a_1 a_2 \cdots a_m/2 \cdot 2 \cdots 2 |^{-2}.$$

Proof. Observe that because $A_i/d_i^p > 0$ for all i, it follows from (4) that $T^k_{m,m+p} > 0$ for all $k \in S(m)$. Hence

$$\begin{aligned} | \Delta_{m-1,m+p} | &= \left| (-1)^{m(m-1)/2} e^{-m(m+p+1)i\theta} \sum_{k \in S(m)} T^k_{m,m+p} \right| \\ &= \sum_{k \in S(m)} T^k_{m,m+p} > 0. \end{aligned}$$

Also

$$\begin{aligned} | \Delta_{m,m+p+1} | &= \sum_{k' \in S(m+1)} T^{k'}_{m+1,m+p+1} \\ &= \sum_{k \in S(m)} \Bigg[T^k_{m,m+p} (d_{k_1} d_{k_2} \cdots d_{k_m})^{-2} \\ &\quad \times \left(\sum_{i=k_m+1}^{\infty} \frac{A_i}{d_i^{p+2}} \right) \left(1 - \frac{d_{k_m}}{d_{k_{m+1}}} \right)^2 \cdots \left(1 - \frac{d_{k_1}}{d_{k_{m+1}}} \right)^2 \Bigg] \\ &\leqslant \left(\sum_{k \in S(m)} T^k_{m,m+p} \right) | a_1 a_2 \cdots a_m |^{-2} \left(\sum_{i=m+1}^{\infty} \frac{A_i}{d_i^{p+2}} \right) 2^{2m} \\ &\leqslant | \Delta_{m-1,m+p} | \, | a_1 a_2 \cdots a_m/2 \cdot 2 \cdots 2 |^{-2} \, | c_{p+1} |. \end{aligned}$$

This proves the lemma.

Let $\tilde{f}(x)$ denote the function obtained from $f(x)$ by setting $\theta = 0$, so that $a_i = d_i$. Let $\tilde{\Delta}_{m-1,m+p}$, $\tilde{\Delta}_{m,m+p}(x)$, $\tilde{P}_{m,m+p}(x)$ and $\tilde{Q}_{m,m+p}(x)$ denote quantities corresponding to $\tilde{f}(x)$. We then have:

LEMMA 2. *The polynomials* $\tilde{Q}_{m,m+p}(x)$ *have only real zeroes.*

Proof. From the identity

$$\begin{aligned} &\tilde{P}_{m+2,m+p+2} \tilde{Q}_{m,m+p} - \tilde{P}_{m,m+p} \tilde{Q}_{m+2,m+p+2} \\ &\quad = [\tilde{f}(x) \, \tilde{Q}_{m,m+p} - \tilde{P}_{m,m+p}] \, \tilde{Q}_{m+2,m+p+2} \\ &\quad\quad - [\tilde{f}(x) \, \tilde{Q}_{m+2,m+p+2} - \tilde{P}_{m+2,m+p+2}] \, \tilde{Q}_{m,m+p} \end{aligned}$$

and (8), we see that the right-hand side is divisible by x^{2m+p+1}. Hence since the left-hand side is a polynomial of degree at most $2m + p + 2$ it must have the form

$$(r'_{m,m+p} + s'_{m,m+p} x) \, x^{2m+p+1}. \tag{11}$$

From (9) we obtain

$$\tilde{P}_{m+1,m+p+1}\tilde{Q}_{m,m+p} - \tilde{P}_{m,m+p}\tilde{Q}_{m+1,m+p+1} = (-1)^m(\tilde{\Delta}_{m,m+p+1}/\tilde{\Delta}_{m-1,m+p})\, x^{2m+p+1}, \tag{12}$$

and

$$\tilde{P}_{m+2,m+p+2}\tilde{Q}_{m+1,m+p+1} - \tilde{P}_{m+1,m+p+1}\tilde{Q}_{m+2,m+p+2} = (-1)^{m+1}(\tilde{\Delta}_{m+1,m+p+1}/\tilde{\Delta}_{m,m+p})\, x^{2m+p+3}.$$

Substituting (11) and (12) into the identity

$$[\tilde{P}_{m+1,m+p+1}\tilde{Q}_{m,m+p} - \tilde{P}_{m,m+p}\tilde{Q}_{m+1,m+p+1}]\, \tilde{Q}_{m+2,m+p+2} + [\tilde{P}_{m,m+p}\tilde{Q}_{m+2,m+p+2} - \tilde{P}_{m+2,m+p+2}\tilde{Q}_{m,m+p}]\, \tilde{Q}_{m+1,m+p+1} = [\tilde{P}_{m+1,m+p+1}\tilde{Q}_{m+2,m+p+2} - \tilde{P}_{m+2,m+p+2}\tilde{Q}_{m+1,m+p+1}]\, \tilde{Q}_{m,m+p}$$

yields the equation

$$(-1)^m \frac{\tilde{\Delta}_{m,m+p+1}}{\tilde{\Delta}_{m-1,m+p}}\, x^{2m+p+1}\tilde{Q}_{m+2,m+p+2} - (r'_{m,m+p} + s'_{m,m+p}x)\, x^{2m+p+1}\tilde{Q}_{m+1,m+p+1} = (-1)^m\, (\tilde{\Delta}_{m+1,m+p+2})/(\tilde{\Delta}_{m,m+p+1})\, x^{2m+p+3}\tilde{Q}_{m,m+p}\,.$$

On dividing this equation by $(-1)^m(\tilde{\Delta}_{m,m+p+1}/\tilde{\Delta}_{m-1,m+p})\, x^{2m+p+1}$ and transposing we get

$$\tilde{Q}_{m+2,m+p+2} = (r''_{m,m+p} + s''_{m,m+p}x)\, \tilde{Q}_{m+1,m+p+1} + (\tilde{\Delta}_{m-1,m+p}\tilde{\Delta}_{m+1,m+p+1}/\tilde{\Delta}^2_{m,m+p+1})\, x^2\tilde{Q}_{m,m+p}\,.$$

Setting $x = 0$ and using the fact that $Q_{m,n}(0) = 1$ it follows that $r''_{m,m+p} = 1$. Then introducing the notation

$$s_{m,m+p} = s''_{m,m+p}\,,$$

$$t_{m,m+p} = -(\tilde{\Delta}_{m-1,m+p}\tilde{\Delta}_{m+1,m+p+1}/\tilde{\Delta}^2_{m,m+p+1}),$$

one gets

$$\tilde{Q}_{m+2,m+p+2} = (1 + s_{m,m+p}x)\, \tilde{Q}_{m+1,m+p+1} - t_{m,m+p}x^2\tilde{Q}_{m,m+p}\,. \tag{13}$$

The numbers $s_{m,m+p}$ and $t_{m,m+p}$ are real and in fact using (4) and (5) it is easy to see that $t_{m,m+p} > 0$. Now consider the new polynomials defined by

$$Q^*_{m,m+p}(x) = x^m\tilde{Q}_{m,m+p}\left(\frac{1}{x}\right). \tag{14}$$

From (13) and (14) it follows that

$$Q^*_{m+2,m+p+2} = (s_{m,m+p} + x)\, Q^*_{m+1,m+p+1} - t_{m,m+p} Q^*_{m,m+p}\,. \tag{15}$$

One now proceeds as in the theory of G-fractions; see [2, p. 204]. Let

$$Q^*_{m,m+p}(\bar{x})\, Q^*_{m+1,m+p+1}(x) = X_{m,m+p} + iY_{m,m+p}\,. \tag{16}$$

Multiplying both sides of (15) by $Q^*_{m+1,m+p+1}(\bar{x})$ and equating imaginary parts we get

$$Y_{m+1,m+p+1} = \eta \mid Q^*_{m+1,m+p+1} \mid^2 + t_{m,m+p} Y_{m,m+p}\,,$$

where $x = \xi + i\eta$. Hence, if $\eta \neq 0$,

$$Y_{m+1,m+p+1}/\eta \geqslant t_{m,m+p}(Y_{m,m+p}/\eta),$$

and since $Y_{0,p} = \eta$ it follows that

$$\frac{Y_{m+1,m+p+1}}{\eta} \geqslant \prod_{i=0}^{m} t_{i,i+p} > 0. \tag{17}$$

From (16) and (17) we see that if $\eta \neq 0$, then $Q^*_{m+1,m+p+1}(x) \neq 0$, and hence the polynomial has only real zeroes. Since the roots of $\tilde{Q}_{m+1,m+p+1}(x)$ are the reciprocals of the roots of $Q^*_{m+1,m+p+1}$, the lemma follows.

LEMMA 3. *The polynomials $Q_{m,m+p}(x)$ have all their zeroes in the set S_θ, and, in fact, for any compact set G, bounded away from S_θ, there exists a $\delta > 0$ such that*

$$| Q_{m,m+p}(x) | \geqslant \delta^m \qquad \textit{for all} \quad x \in G.$$

Proof. From (4)–(7) it follows easily that

$$\Delta_{m-1,m+p} = \exp(-m(m+p+1)\, i\theta)\, \tilde{\Delta}_{m-1,m+p}\,,$$

and

$$\Delta_{m,m+p}(x) = \exp(-m(m+p+1)\, i\theta)\, \tilde{\Delta}_{m,m+p}(xe^{-i\theta}),$$

so that

$$Q_{m,m+p}(x) = \tilde{Q}_{m,m+p}(xe^{-i\theta}).$$

It is therefore sufficient to prove the lemma in the case $\theta = 0$. Hence we consider the function $\tilde{f}(x)$ introduced above and we assume that G is bounded away from S_0.

For $0 < \delta_1 < |a_1|$, define

$$G_1(\delta_1) = \{x: |\operatorname{Im}(x)| \geqslant \delta_1\},$$
$$G_2(\delta_1) = \{x: |\operatorname{Re}(x)| \leqslant |a_1| - \delta_1\},$$
$$G_3(\delta_1) = \{x: \operatorname{Re}(x) \leqslant |a_1| - \delta_1\}.$$

Now since G is bounded away from S_0, we may choose $\delta_1 > 0$ such that

(18) $$G \subseteq G_1(\delta_1) \cup G_3(\delta_1) \quad \text{in case A, and}$$
$$G \subseteq G_1(\delta_1) \cup G_2(\delta_1) \quad \text{in cases B and C.}$$

Since G is bounded, choose R such that $G \subseteq \{x: |x| \leqslant R\}$. We claim that $\delta = \delta_1/R$ satisfies the requirements of the lemma. To show this, let $u_1, u_2, \ldots u_m$ denote the roots of $\tilde{Q}_{m,m+p}(x)$. From (14) we have

$$|\tilde{Q}_{m,m+p}(x)| = |x|^m \left| Q^*_{m,m+p}\left(\frac{1}{x}\right)\right| = |x|^m \prod_{i=1}^{m} \left|\frac{1}{x} - \frac{1}{u_i}\right|$$
$$\geqslant |x|^m \prod_{i=1}^{m} \left|\operatorname{Im}\left(\frac{1}{x} - \frac{1}{u_i}\right)\right| = |x|^m \prod_{i=1}^{m} \left|\operatorname{Im}\left(\frac{1}{x}\right)\right|$$
$$= |x|^m \left(\frac{|\eta|}{|x|^2}\right)^m = \left|\frac{\eta}{x}\right|^m.$$

Therefore for $x \in G_1(\delta_1) \cap G$ we have

(19) $$|Q_{m,m+p}(x)| \geqslant |\eta/x|^m \geqslant (\delta_1/R)^m = \delta^m.$$

Furthermore, since $|1 - (x/u_i)| \geqslant |1 - (\xi/u_i)|$ we get

$$|\tilde{Q}_{m,m+p}(x)| = \prod_{i=1}^{m} |1 - (x/u_i)| \geqslant \prod_{i=1}^{m} |1 - (\xi/u_i)| = |\tilde{Q}_{m,m+p}(\xi)|.$$

Assume $x \in G_2(\delta_1)$. Then

$$\xi/a_i \leqslant |\xi/a_i| \leqslant |\xi/a_1| \leqslant |a_1| - \delta_1)/|a_1| = 1 - \delta_1/|a_1|$$

so that

$$1 - \xi/a_i \geqslant \delta_1/|a_1| \geqslant \delta_1/R = \delta.$$

Hence for all $k \in S(m)$,

$$\prod_{\nu=1}^{m} (1 - \xi/a_{k_\nu}) \geqslant \delta^m.$$

It therefore follows from (5)–(7) and the fact that $T^k_{m,m+p} > 0$ that

$$|\tilde{Q}_{m,m+p}(x)| \geqslant |\tilde{Q}_{m,m+p}(\xi)| \geqslant \delta_m . \tag{20}$$

The lemma now follows in cases B and C by (18)–(20).

In case A, $a_i > 0$ for all i so that for $x \in G_3(\delta_1)$,

$$\xi/a_i \leqslant (a_1 - \delta_1)/a_1 = 1 - \delta_1/a_1 ,$$

and hence $1 - \xi/a_i \geqslant \delta_1/a_1 \geqslant \delta_1/R = \delta$. Consequently we get, as before, $|\tilde{Q}_{m,m+p}(x)| \geqslant |\tilde{Q}_{m,m+p}(\xi)| \geqslant \delta^m$. This together with (18) proves the lemma in case A as well.

Proof of the Theorem. Let G be as in the last lemma. Hence $G \subseteq \{x: |x| \leqslant R\}$ and there exists $\delta > 0$ such that $|Q_{i,i+p}(x)| \geqslant \delta^i$ for all $x \in G$ and for all i. Choose N such that

$$|a_N| > 4R/\delta,$$

and so

$$|2R/a_i\delta| < \tfrac{1}{2} \qquad \text{for} \quad i \geqslant N.$$

By Lemma 1,

$$\left| (-1)^i \frac{\Delta_{i,i+p+1}}{\Delta_{i-1,i+p}} \frac{x^{2i+p+1}}{Q_{i,i+p}Q_{i+1,i+p+1}} \right|$$
$$\leqslant |c_{p+1}| \left| \frac{a_1a_2\cdots a_i}{2\cdot 2\cdots 2} \right|^{-2} \frac{R^{2i+p+1}}{\delta^i\,\delta^{i+1}} = |c_{p+1}| \frac{R^{p+1}}{\delta} \left| \frac{2R}{a_1\,\delta}\frac{2R}{a_2\,\delta}\cdots\frac{2R}{a_i\,\delta} \right|^2$$
$$\leqslant |c_{p+1}| \frac{R^{p+1}}{\delta} \left| \frac{2R}{a_1\,\delta}\cdots\frac{2R}{a_N\,\delta} \right|^2 \left(\frac{1}{2^{i-N}}\right)^2 \leqslant M4^{-i}, \qquad \text{for all} \quad x \in G.$$

It follows that the sum (10) converges uniformly to a holomorphic function $g(x)$ in G. Since we also have uniform convergence in a neighborhood of the origin and since $f^{(n)}_{m,m+p}(0) = n!c_n$ for $n < 2m + 1$ we see that $g^{(n)}_{(0)} = n!c_n$. Hence $g(x) = f(x)$ for all $x \in G$. This completes the proof of the theorem.

EXAMPLES. Some simple examples to which our theorem applies are: $\tan x$, $\cot x - 1/x$, $[\Gamma'(x)/\Gamma(x)] + (1/x)$, $\tanh x$ and $\coth x$.

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